PHYSICAL REVIEW E, VOLUME 63, 036128

Self-generated power-law tails in probability distributions

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(Received 6 September 2000; published 27 February 2001)

We consider random processes characterized by the presence of correlations in their variance, or more generally in some of their moments. Typical examples are constituted by autoregressive conditional heteroskedasticity (ARCH) processes which are known to display power-law tails in the associated probability distributions. Here, we determine the corresponding exponents exactly and extend these results to relaxation phenomena which can be expected to play a role in natural sciences.

DOI: 10.1103/PhysRevE.63.036128 PACS number(s): 02.50.Ey, 05.40.Fb, 87.23.Ge, 89.90.+n

Probability distribution functions displaying power-law tails are ubiquitous in natural sciences and finance, at least in some range of length or time scales [1]. Their study is clearly an important task for undertanding the underlying mechanisms that generate them in the context of natural phenomena [2] and in the description of the distribution of stock prices in finance [3].

Here, we study ''self-regulated'' random processes displaying correlations in the moments of their standard deviation, as a simple generalization of the well-known autoregressive conditional heteroskedasticity (ARCH) processes, in which the correlations are introduced in their variance [4]. Although it is known that the associated probability distribution functions (PDFs) decay asymptotically as power laws (cf. e.g., Podobnik *et al.* in Ref. [3]), the exact value of the corresponding asymptotic exponents are known in a particular case only [5]. The aim of our work is to determine them in a more general fashion. Such relations are not only interesting from a theoretical point of view, but might also prove useful concerning applications, for example, in the context of extremal events [5].

The random processes we are interested in, denoted generically as $\mathcal{R}$, produce a sequence of numbers $\{x_n\}$ according to a recursion relation of the form

$$x_n = \mathcal{R}(x_{n-1}), \tag{1}$$

where the outcome x_n after the nth iteration step depends only on its previous value x_{n-1} after the $(n-1)$th step, reflecting the Markov character of the process. The random process $\mathcal{R}(y)$ in Eq. (1) is completely specified by giving the transition probability $W(y \curvearrowright x)dx$, representing the probability that starting from the value y the outcome will fall into the interval $[x,x+dx]$.

The transition PDF $W(y \curvearrowright x)$ is assumed, similarly as for ARCH(1) processes [4], to be given by

$$W(y \curvearrowright x) \equiv \frac{1}{\sqrt{2\pi}\sigma_y} \exp\left(-\frac{x^2}{2\sigma_y^2}\right), \tag{2}$$

where here

$$\sigma_y^2 \equiv (a + b|y|^q)^{2/q}, \tag{3}$$

and $q>0$, $a>0$, and $b \geqslant 0$ are the model parameters, and the prefactor $1/(\sqrt{2\pi}\sigma_y)$ ensures the normalization condition $\int_{-\infty}^{\infty} dx W(y \curvearrowright x) = 1$. The case $q=2$ corresponds to standard ARCH(1) processes, while the linear form for the case $q=1$ corresponds to the so-called linear ARCH(1), or LARCH(1) processes [6].

A simple prescription can be derived for dealing with Eq. (1) numerically. To do this, note that the integral $I(y,x) \equiv \int_{-\infty}^{x} dx' W(y \curvearrowright x')$, where $I(y,x) \in [0,1]$ $\forall x,y$, is the accumulated probability that the transition occurs from y into the interval $(-\infty,x]$. Hence to obtain a value x_n after the nth step we generate an uncorrelated random number r_n, homogenously distributed in the interval $[0,1]$, and determine x_n such that $I(x_{n-1},x_n) = r_n$ is satisfied. Solving for x_n, we obtain

$$x_n = \mathcal{R}(x_{n-1}) \equiv \sqrt{2}\sigma_{x_{n-1}} \mathrm{erf}^{-1}(2r_n - 1), \tag{4}$$

where $\sigma_{x_{n-1}} = (a + b|x_{n-1}|^q)^{1/q}$ and $\mathrm{erf}^{-1}(z)$ denotes the inverse error function. The relation, Eq. (4), can be used to obtain the corresponding PDF $P(x)$ numerically, where $P(x)dx$ denotes the probability to find a value between x and $x+dx$ after a given iteration.

It is possible to make a step forward by determining the equation that $P(x)$ satisfies. To this end, note that the probability $P(x)dx$ is the result of all possible transitions $y \curvearrowright x$ from an interval $[y,y+dy]$ into the interval $[x,x+dx]$ during a given iteration step. Similarly, the probability to be in the interval $[y,y+dy]$ after the previous step is given by the PDF $P(y)$ itself. Therefore the PDF $P(x)$ for the random processes in Eq. (4) satisfies the self-consistent equation

$$P(x) = \int_{-\infty}^{\infty} dy W(y \curvearrowright x) P(y). \tag{5}$$

Although a complete analytical solution of Eq. (5) does not seem feasible, one can argue that a power-law decay

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$$P(x) \sim |x|^{-\alpha} \quad \text{for} \quad |x| \to \infty \tag{6}$$

is consistent with Eq. (5). To see this we use Eq. (2) in Eq. (5) and note that for $|x| \to \infty$ the main contribution to the integral comes from asymptotically large values of $|y|$. This suggests us that one can accurately replace σ_y by its limiting expression $\sigma_y = b^{1/q}|y|$ (i.e., when $b|y|^q \gg a$) and $P(y)$ by $|y|^{-\alpha}$ in the integrand of Eq. (5), to obtain

$$|x|^{-\alpha} = \frac{2}{\sqrt{2\pi b^{2/q}}} \int_0^\infty dy\, y^{-\alpha-1} \exp\left(-\frac{x^2}{2b^{2/q}y^2} \right). \tag{7}$$

The integral is finite for $\alpha > 0$ and can be evaluated exactly yielding

$$|x|^{-\alpha} = \left[\frac{1}{\sqrt{\pi}} (2b^{2/q})^{(\alpha-1)/2} \Gamma(\alpha/2) \right] |x|^{-\alpha}. \tag{8}$$

The quantity within parenthesis on the right-hand side of the equation must equal 1 in order to be consistent with the left-hand side of the equation. Thus the following exact relation between the parameter b and the exponent α is obtained,

$$b^{-(\alpha-1)/q} = \frac{1}{\sqrt{\pi}} 2^{(\alpha-1)/2} \Gamma(\alpha/2), \tag{9}$$

consistent with the known result for $q=2$ [5]. In general Eq. (9) cannot be solved for α explicitly, but can be used to determine the value of b required to yield a given value of the exponent α.

It is therefore instructive to study the limiting cases $b \to 0$ and $b \to \infty$ from Eq. (9), for which explicit relations can be derived. In the case $b \to 0$, one expects that $\alpha \to \infty$, since $P(x)$ reduces to a Gaussian in that limit [cf. Eq. (5)]. Using the asymptotic form $\Gamma(z) \cong (z/e)^z \sqrt{2\pi z}$, when $z \to \infty$ (see, e.g., Ref. [7]), we find

$$\alpha = \frac{e}{b^{2/q}} \quad \text{for} \quad b \to 0. \tag{10}$$

In the case $b \to \infty$, we expect that $\alpha \to 0$. Using then $\Gamma(z) \cong 1/z$, when $z \to 0$, we obtain

$$\alpha = \sqrt{\frac{2}{\pi}} \frac{1}{b^{1/q}} \quad \text{for} \quad b \to \infty. \tag{11}$$

The value of the exponent α is intimately related to the moments M_p of the PDF $P(x)$, $M_p \equiv \langle |x|^p \rangle = \int_{-\infty}^{\infty} dx |x|^p P(x)$ (with $p > -1$ to ensure integrability around the origin), which diverge for $p \geq \alpha - 1$. Clearly, $P(x)$ can be normalized according to $\int_{-\infty}^{\infty} dx P(x) = 1$, when $\alpha > 1$. The corresponding values of b can be found easily by taking the limit $\alpha \to 1$ in Eq. (9). Thus values $\alpha \geq 1$ correspond to

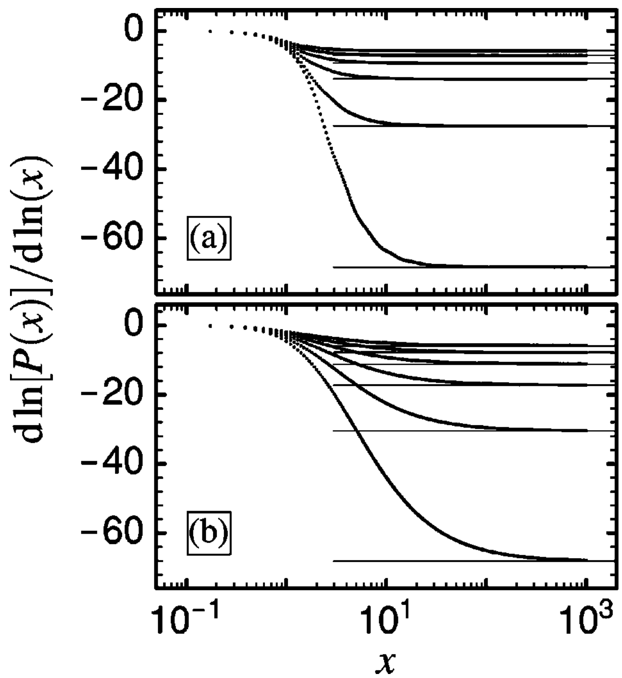

FIG. 1. Successive slopes $d \ln[P(x)]/d \ln(x)$ of the PDF $P(x)$ versus x for the cases: (a) $q=2$ (ARCH), with $a=0.18$ for (from bottom to top) $b=0.04$, 0.1, 0.2, 0.3, 0.4, and 0.5 (continuous lines). (b) $q=1$ (LARCH), with $a=0.379$ for (from bottom to top) $b=0.2$, 0.3, 0.4, 0.5, 0.6, and 0.7 (continuous lines). The continuous lines denote the results obtained by numerically solving Eq. (5), and the horizontal straight lines the asymptotic values expected for α from Eq. (9).

$$b \leq b_0 \equiv 2 \exp\left[\frac{q}{2\sqrt{\pi}} \left(2 - \sqrt{\pi} + \frac{\gamma}{\ln 2} \right) \right], \tag{12}$$

where $\gamma \cong 0.5772156649$ is the Euler's constant. Equation (12) yields $b_0 \cong 3.6377$ for $q=2$, and $b_0 \cong 2.6973$ for $q=1$.

A prominent role is also played by the second moment (or variance, σ^2), and the fourth moment which is related to the kurtosis $\kappa \equiv \langle x^4 \rangle / \sigma^4$. The variance diverges when $\alpha \leq 3$ and the kurtosis when $\alpha \leq 5$. For instance, for $q=2$, Eq. (9) yields $b=1$ when $\alpha=3$, corresponding to the divergence of σ^2 according to the known result $\sigma^2 = a/(1-b)$ [4]. Similarly, κ diverges when $\alpha=5$, corresponding to the value $b = 1/\sqrt{3}$, consistent with the exact expression $\kappa = 3 + 6b^2/(1-3b^2)$ [4]. In the case of LARCH, i.e., for $q=1$, the variance is given by

$$\sigma^2 = \frac{a^2 d_2}{1-b^2} \tag{13}$$

with $d_2 \equiv 1 + 2c_1 b/(1-c_1 b)$ and $c_1 = \sqrt{2/\pi}$, while the fourth moment reads

$$\langle x^4 \rangle = \frac{3a^4}{1-3b^4} \left[2d_2 \frac{1+2b^2}{1-b^2} + \frac{8c_1 b^3 d_3}{1-2c_1 b^3} - 1 \right] \tag{14}$$

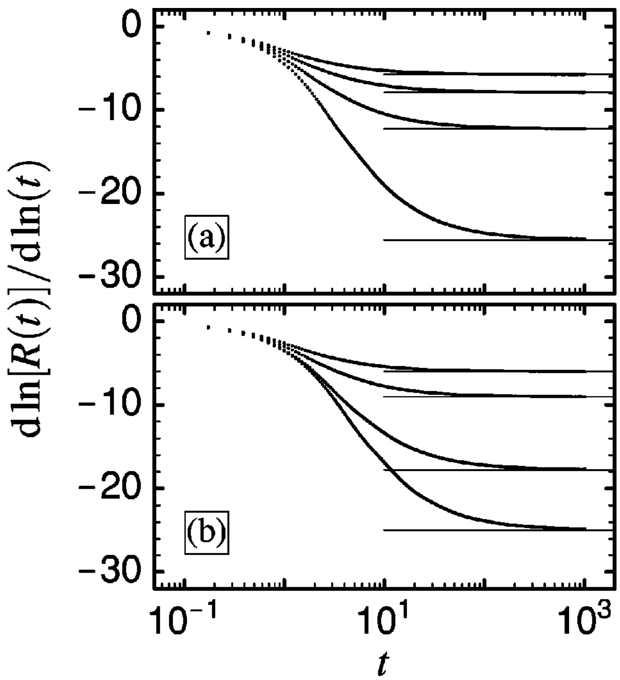

FIG. 2. Successive slopes $d\ln[R(t)]/d\ln(t)$ for the relaxation function $R(t)$ versus t for the cases: (a) $\beta=1$, with $\tau_0=0.2$ for (from bottom to top) $a=0.1$, 0.2, 0.3, and 0.4 (continuous lines). (b) $\beta=0.9$ with $\tau_0=0.2$ for (from bottom to top) $a=0.07$, 0.1, 0.2, and 0.3 (continuous lines). The continuous lines denote the results obtained by numerically solving Eq. (15), using Eq. (16), and the straight lines the asymptotic values expected for α from Eq. (17).

with $d_3\equiv 1+3c_1b/(1-c_1b)+3d_2b^2/(1-b^2)$, indicating that σ^2 diverges again for $b=1$, while the kurtosis for $b=3^{-1/4}$, as can be easily verified from Eq. (9). Numerical results for the case $q=2$ and $q=1$ shown in Fig. 1 confirm Eq. (9). Note the rather slow convergence of $d\ln[P(x)]/d\ln(x)$ towards its asymptotic value.

Let us next consider relaxation phenomena. Within the present context, we aim to describe a relaxation process $R(t)$, being a function of say, the time t, by a self-consistent equation similar to Eq. (5),

$$R(t)=\int_0^\infty d\tau W(\tau,t)R(\tau). \tag{15}$$

The process $R(t)$ can be seen as resulting from a ''self-regulated'' superposition of different relaxation channels, each one characterized by a time decay of the form

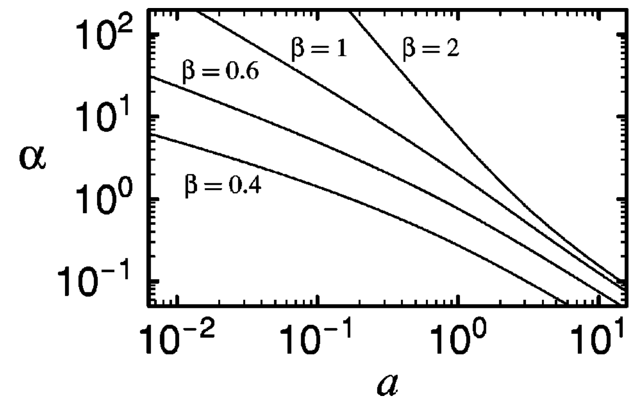

FIG. 3. Power-law relaxation exponent α versus model parameter a, after Eq. (17), for different values of the exponent β.

$$W(\tau,t)=\frac{1}{A(\beta)\sigma_\tau}\exp\left(-\left[\frac{t}{\sigma_\tau}\right]^\beta\right), \tag{16}$$

where $\sigma_\tau=\tau_0+a\tau$, $\tau_0>0$, $0\leqslant a<\infty$, and $A(\beta)=\Gamma(1/\beta)/\beta$, with $\beta>0$. Simple exponential relaxation of $W(\tau,t)$ corresponds to $\beta=1$, for which $A(1)=1$. Clearly, in the case of a single relaxation channel, i.e., for $a=0$, $R(t)=[A(\beta)\tau_0]^{-1}\exp[-(t/\tau_0)^\beta]$.

Here again, we are interested in the asymptotic form of $R(t)$ when $t\to\infty$. Following the same procedure outlined above for generalized ARCH processes, we find that $R(t)\sim t^{-\alpha}$, when $t\to\infty$, where the exponent α obeys the relation,

$$a^{-(\alpha-1)}=\frac{\Gamma(\alpha/\beta)}{\Gamma(1/\beta)}. \tag{17}$$

Numerical results for the cases $\beta=1$ and $\beta=0.9$ shown in Fig. 2 confirm Eq. (17).

Regarding the limiting behaviors of α, we find $\alpha=e\beta/a^\beta$ when $a\to0$, and $\alpha=\beta/[a\Gamma(1/\beta)]$ when $a\to\infty$. The case $\alpha\geqslant1$ corresponds to the values $a\leqslant a_1\equiv\exp[-\Gamma'(1/\beta)/\Gamma(1/\beta)]$, where $\Gamma'(x)$ denotes the first derivative of $\Gamma(x)$ with respect to x. For instance, for $\beta=1$ one has $a_1=\exp(\gamma)\cong1.7811$. The behavior of α versus the parameter a is shown in Fig. 3 for selected values of the exponent β.

M.P. gratefully acknowledges the Alexander von Humboldt Foundation (Feodor Lynen program) for partial financial support.

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